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CALIFORNIA UNIV LOS ANGELES DEPT OF ENGINEERING SYSTEMS F/G 12/1
AN OPTIMAL LINEAR TIME INVARIANT ESTIMATOR FOR CERTAIN TYPES OF--ETC(U)
1976 T L GREENLEE, C T LEONDES AF-AFOSR-2958-76

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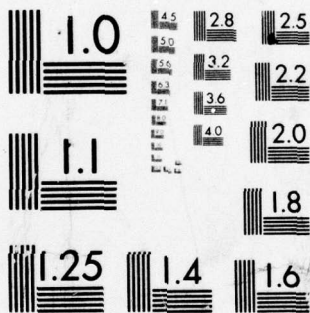


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Abstract

A technique is developed whereby one can synthesize a causal, linear time-invariant estimator that is optimal for restricted types of nonstationary processes. The technique is applicable to linear, time-invariant systems (driven by nonstationary state noise) for which scalar observations are made in the presence of additive nonstationary noise. Two-dimensional Fourier transforms are used to obtain an expression for the estimator's mean square error. It is assumed that it is desirable to minimize the time integral of this expression. The calculation of this integral results in an expression which can be minimized by selecting an estimator depending in a prescribed way on the two-dimensional Fourier transforms of the state and observation noise. The resulting estimator is causal, linear, and time invariant. It is similar in some respects to the Wiener filter that can be derived under the assumptions of stationary state and observation noise processes. The estimator's usefulness is limited by the requirement that the observations be scalar, and the nonstationary processes have Fourier transformable autocorrelation functions.¹

I. Introduction

The determination of optimal linear time invariant estimators that can be used to process the output of a linear time invariant system driven by stationary noise and observed in the presence of additive stationary noise, was the subject of Wiener's work [1]. This early work was followed by Kalman's [2] determination of a method for finding optimal estimates given certain types of nonstationary disturbances and corruptions. A major distinction between these two results was the methods by which they were obtained. Wiener's method was frequency domain based, while Kalman's approach relied on time domain techniques. A survey of pertinent literature indicates that until Kalman's results were published, the major work on nonstationary processes was confined to looking at slight modifications of the stationary problem.² When Kalman's results were announced, there was a sudden transition from frequency domain methods and little attention has been paid to frequency domain techniques in estimation since that time.

In many applications, the Kalman estimator results in exorbitant computational requirements.

This is due mostly to the fact that its time varying covariance matrix must be calculated before the estimator gains can be computed. In an effort to make the estimator less complex, most designers use the steady state gain values or schedule sets of gain values that are to be used at different points in the processing algorithm. This results in a simpler estimator that performs suboptimally during periods when the process being observed is nonstationary. The estimator will be optimal when the process becomes stationary provided the steady state value of the estimation error covariance is used in the gain computation.

Use of the stationary gain values constitutes selection of the Wiener estimator as a special case of the Kalman estimator. It is this procedure for selecting a suboptimal estimator that prompts one to ask if there might be a causal, linear, time-invariant estimator that is optimal in some restricted sense for some class of nonstationary processes.

To arrive at such an estimator, it seems reasonable to consider the case most likely to yield such a result. Namely, the system or state model should be linear and time invariant. It should be driven by zero mean nonstationary noise and observed in the presence of additive nonstationary noise. Optimality, for the sake of obtaining results comparable to Wiener and Kalman, should be defined in the sense that the mean-square estimation error or some function of it is minimized. If one returns to frequency domain considerations, it might be possible to arrive at an estimator via techniques paralleling those of Wiener, provided Fourier transforms of the noise processes can be defined. This leads to the use of two-dimensional Fourier transforms to obtain the average energy spectral densities of the state and observation noise processes. An expression is arrived at for the mean-square estimation error and it is assumed that the optimal estimate should minimize the doubly infinite time integral of this error. The expression that results from doing this is used to determine the structure of the optimal estimator. The resultant estimator is similar in some respects to the Wiener estimator. The estimator's usefulness is limited by the requirement that the

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²See reference 12.

Paper , Decision and Control Conference, Clearwater Beach, Florida, 1976.
Proceedings

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observations be scalar and the noise processes have Fourier transformable autocorrelation functions that can successfully undergo spectral factorization.

II. System Assumptions and Outline of Approach

The solution of the problem posed in the introduction is developed in four steps. These are:

1. Consider a linear, time-invariant system driven by nonstationary noise and observed in the presence of additive nonstationary noise. Assume the observations are scalar.
2. Formulate an expression for the mean square estimation error of a linear time-invariant estimator with impulse response $h(\tau)$, that might be used to obtain an estimate of the noise-free observation.
3. The expression for the mean square estimation error obtained in step 2 will in general be time dependent due to the noise being nonstationary. Provided that one is willing to accept a filter that minimizes this expression in an average sense, i.e., over a doubly infinite time interval, one can arrive at a simpler expression for the integral of the mean square error that does not depend on time.
4. Finally, if one restricts the type of nonstationary noise processes sufficiently, the expression obtained in step 3 can be minimized by a properly designed causal, linear, time-invariant estimator that minimizes the integral of the mean square estimation error.

The following sections will expand and elaborate on the four steps. A final section will discuss the implications of the final result.

III. System and Noise Models

Proceeding with the problem solution, consider the error propagation. The system being considered is linear and time invariant. As such, it is described by the following linear, time-invariant vector matrix differential equations:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) \quad (3.1)$$

$$\underline{y}(t) = \underline{C}\underline{x}(t) + \underline{D}\underline{v}(t) = s(t) + n(t)$$

where

$\underline{x}(t)$: is a vector representing the state of the system

$\underline{y}(t)$: is a vector representing an observation of the state of the system (scalar)

$\underline{u}(t), \underline{v}(t)$: are random processes

The problem is to estimate any time-invariant (but not necessarily causal) linear functional $w(t)$ of the signal $s(t)$ defined by the following infinite convolution integral:

$$w(t) = \int_{-\infty}^{\infty} i(\lambda) s(t - \lambda) d\lambda \quad (3.2)$$

It is desired to determine, for every t , an estimate $\hat{w}(t)$, for $w(t)$ which minimizes

$$E[w(t) - \hat{w}(t)]^2 \quad (3.3)$$

or some function of this expression by processing only the "present and past" values of the observed data:

$$y(\tau) = s(\tau) + n(\tau) \quad -\infty \leq \tau \leq t$$

It is further required that

$$\hat{w}(t) = \int_{-\infty}^{\infty} h(\tau) y(t-\tau) d\tau; \quad h(\tau) = 0, \tau < 0 \quad (3.4)$$

In preparation for arriving at an expression for the estimator's mean square estimation error let us introduce the two-dimensional Fourier transform.

Definition: The two-dimensional Fourier transform of a function $f(t_1, t_2)$ is given by:²

$$F(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2) e^{-j(\omega_1 t_1 + \omega_2 t_2)} dt_1 dt_2$$

its inverse is defined by

$$f(t_1, t_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(j\omega_1, j\omega_2) e^{j(\omega_1 t_1 + \omega_2 t_2)} d\omega_1 d\omega_2$$

As in the case of one-dimensional transforms, these two expressions constitute a Fourier transform pair that we will represent symbolically as

$$f(t_1, t_2) \rightleftharpoons F(j\omega_1, j\omega_2)$$

It should be noted that if $f(t_1, t_2)$ is such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t_1, t_2)| dt_1 dt_2 < \infty$$

then $F(j\omega_1, j\omega_2)$ exists for every ω_1, ω_2 and it is bounded.

²For a description of two-dimensional Fourier transforms and their application to nonstationary processes, refer to Papoulis, A., Probability, Random Variables, and Stochastic Processes. McGraw-Hill, 1965, Chapter 12.

IV. The Mean Square Error Expression

We can compute an expression for $e(t) = \omega(t) - \hat{\omega}(t)$. To do this recognize that:

$$\hat{\omega}(t) = \int_{-\infty}^{\infty} y(t-\tau) h(\tau) d\tau \quad (4.1)$$

and for the present let

$$\omega(t) = \int_{-\infty}^{\infty} s(t-\tau) i(\tau) d\tau \quad (4.2)$$

then by subtraction

$$e(t) = \omega(t) - \hat{\omega}(t)$$

and the autocorrelation of $e(t)$ is

$$R_{ee}(t_1, t_2) = E\{e(t_1)e(t_2)\} = E\{[\omega(t_1) - \hat{\omega}(t_1)][\omega(t_2) - \hat{\omega}(t_2)]\} \quad (4.3)$$

Substitution, expansion, and the interchange of expectation and integration yields:

$$\begin{aligned} R_{ee}(t_1, t_2) &= R_{ss}(t_1, t_2) * i(t_1) * i(t_2) \\ &\quad - R_{ss}(t_1, t_2) * i(t_1) * h(t_2) \\ &\quad - R_{ss}(t_1, t_2) * h(t_1) * i(t_2) \\ &\quad + R_{ss}(t_1, t_2) * h(t_1) * h(t_2) \\ &\quad + R_{nn}(t_1, t_2) * h(t_1) * h(t_2) \end{aligned} \quad (4.4)$$

Where we have assumed terms in $R_{sn}(t_1, t_2)$ and $R_{ns}(t_1, t_2)$ are zero and "*" denotes convolution in the time domain.³ Assume each of the above correlation functions has a two-dimensional transform defined as follows:

$$\begin{aligned} R_{ss}(t_1, t_2) &\xleftrightarrow[t_2]{t_1} \Gamma_{ss}(\omega_1, \omega_2) \\ R_{nn}(t_1, t_2) &\xleftrightarrow[t_2]{t_1} \Gamma_{nn}(\omega_1, \omega_2) \end{aligned} \quad (4.5)$$

Next we make use of a property from [3] page 442.

$$R(t_1, t_2) * W(t_1) * W(t_2) \xleftrightarrow[t_2]{t_1} \Gamma(\omega_1, \omega_2) W(\omega_1) W(\omega_2) \quad (4.6)$$

with

$$\begin{aligned} W(\omega_1) &\triangleq \int_{-\infty}^{\infty} w(t_1) e^{-j\omega_1 t_1} dt_1 = W(j\omega_1) \\ W(\omega_2) &\triangleq \int_{-\infty}^{\infty} w(t_2) e^{-j\omega_2 t_2} dt_2 = W(j\omega_2) \end{aligned} \quad (4.7)$$

being the standard one-dimensional Fourier transforms of $w(t_1)$ and $w(t_2)$ respectively.

Taking the two-dimensional Fourier transform of $R_{ee}(t_1, t_2)$ we have:

$$\begin{aligned} \Gamma_{ee}(\omega_1, \omega_2) &= \Gamma_{ss}(\omega_1, \omega_2) \left[I(j\omega_1) - H(j\omega_1) \right] \\ &\quad \left[I(j\omega_2) - H(j\omega_2) \right] + \Gamma_{nn}(\omega_1, \omega_2) \\ &\quad \left[H(j\omega_1) \right] \left[H(j\omega_2) \right] \end{aligned} \quad (4.8)$$

If we now take the inverse transform of $\Gamma_{ee}(\omega_1, \omega_2)$, we have an expression for the autocorrelation of the estimation error:

$$\begin{aligned} R_{ee}(t_1, t_2) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \Gamma_{ss}(\omega_1, \omega_2) \right. \\ &\quad \left[I(j\omega_1) - H(j\omega_1) \right] \left[I(j\omega_2) - H(j\omega_2) \right] \\ &\quad e^{j(\omega_1 t_1 + \omega_2 t_2)} + \Gamma_{nn}(\omega_1, \omega_2) \left[H(j\omega_1) \right] \\ &\quad \left. \left[H(j\omega_2) \right] e^{j(\omega_1 t_1 + \omega_2 t_2)} \right\} d\omega_1 d\omega_2 \end{aligned} \quad (4.9)$$

Recall that we seek an expression for the mean square estimation error for each point in time. Clearly, if $t_1 = t_2 = t$ in the previous expression we have:

$$\begin{aligned} R_{ee}(t, t) &= E\{e(t)e(t)\} \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \Gamma_{ss}(\omega_1, \omega_2) \left[I(j\omega_1) - H(j\omega_1) \right] \right. \\ &\quad \left[I(j\omega_2) - H(j\omega_2) \right] e^{j(\omega_1 + \omega_2)t} + \Gamma_{nn}(\omega_1, \omega_2) \\ &\quad \left. \left[H(j\omega_1) \right] \left[H(j\omega_2) \right] e^{j(\omega_1 + \omega_2)t} \right\} d\omega_1 d\omega_2 \end{aligned} \quad (4.10)$$

³The cross terms can be carried through the development but they provide no additional problems and therefore have been deleted.

Note that this expression for the mean square estimation error is completely general except for the restriction that the indicated signal and noise two-dimensional transforms exist.

If we ponder this expression for a moment, we see that to select an estimator $H(j\omega)$ that would minimize it for each value of t , is not a simple task. In fact, it is not obvious that such an estimator even exists.

It is interesting to note that the assumption of stationary disturbances gives rise to two-dimensional signal and noise transforms that cause the mean square error expression to become independent of time. As shown in [3], the two-dimensional transforms of stationary processes do not exist unless impulses are included in the transforms. When such transforms are substituted in (4.10), the right hand side of this expression yields an integral that is independent of time. This integral is minimized by the causal, linear, time-invariant filter arrived at by N. Wiener. The signal and noise terms in this integral will be represented by the ordinary power spectral densities instead of the average energy spectral densities which will be defined in Section V.

V. Consideration of the Integral of the Mean Square Estimation Error

Consider equation (4.10). We are concerned with selecting an estimator $H(j\omega)$ that will minimize this expression or some function of it. Suppose we address this problem directly by trying to minimize (4.10) for each value of t . If we make the change of variable

$$\omega_1 + \omega_2 = p$$

and hold ω_1 constant while integrating with respect to p , equation (4.10) becomes

$$R_{ee}(t, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_{ss}(\omega_1, p - \omega_1) \right. \\ \left. [I(j\omega_1) - H(j\omega_1)] [I(j(p - \omega_1)) - H(j(p - \omega_1))] \right. \\ \left. + \Gamma_{nn}(\omega_1, p - \omega_1) [H(j\omega_1)] [H(j(p - \omega_1))] \right. \\ \left. d\omega_1 \right\} e^{jpt} dp \quad (5.1)$$

We must now find $H(j\omega)$ such that this expression or some function of $R_{ee}(t, t)$ is minimized for each value of t .

Note that the bracketed expression is an integral with respect to ω_1 and it will result in a function of p only. If we denote this function of p by $E^2(p)$, then (5.1) becomes:

$$R_{ee}(t, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E^2(p) e^{jpt} dp \quad (5.2)$$

From (5.2) we immediately recognize $E_{ee}(t, t) \triangleq E\{e^2(t)\}$ and $E^2(p)$ as a Fourier transform pair. We denote this by:

$$E\{e^2(t)\} \triangleq E^2(p)$$

Using this notation, we also recall:

$$E^2(p) = \int_{-\infty}^{\infty} E\{e^2(t)\} e^{-jpt} dt \quad (5.3)$$

In particular, for $p = 0$ we have the simple result:

$$\int_{-\infty}^{\infty} E\{e^2(t)\} dt = E^2(p) \Big|_{p=0} \quad (5.4)$$

Note that (5.4) implies the doubly infinite time integral of the mean square estimation error is finite provided $E^2(p) \Big|_{p=0}$ is finite. From (5.1)

$$E^2(p) \Big|_{p=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \Gamma_{ss}(\omega_1, -\omega_1) [|I(j\omega_1) - H(j\omega_1)|^2] \right. \\ \left. + \Gamma_{nn}(\omega_1, -\omega_1) [|H(j\omega_1)|^2] \right\} d\omega_1 \quad (5.5)$$

The terms $\Gamma_{ss}(\omega_1, -\omega_1)$ and $\Gamma_{nn}(\omega_1, -\omega_1)$ are called the average energy spectral densities of the signal and noise processes. The right-hand side of (5.5) bears a striking resemblance to the mean square error expression of a Wiener filter. This similarity immediately suggests that one may be able to select a causal, linear, time-invariant estimator $H(j\omega)$, that will minimize (5.5) and thereby the integral of the mean square estimation error. The resulting estimator may be different from the Wiener estimator since the position of the power spectral density has been assumed by the energy spectral density.

VI. The Optimal Linear Time-Invariant Estimator for a Restricted Class of Nonstationary Processes

From the previous section we note that equation (5.4) and (5.5) yield:

$$\int_{-\infty}^{\infty} E\{e^2(t)\} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \Gamma_{ss}(\omega, -\omega) \right. \\ \left. [|I(j\omega) - H(j\omega)|^2] + \Gamma_{nn}(\omega, -\omega) \right. \\ \left. [|H(j\omega)|^2] \right\} d\omega \quad (6.1)$$

^aIn addition, if we assume that the estimate is unbiased, then we can use the terms mean square error and error variance interchangeably.

Assuming this integral exists, the first question one asks is whether or not there is an $H(j\omega)$ for which (6.1) is minimized. This question is answered for a restricted class of processes by the following theorem, which applies to the filtering case ($I(j\omega) = 1$).

Theorem

Suppose that:

- The average energy spectral densities $\Gamma_{ss}(\omega, -\omega)$ and $\Gamma_{nn}(\omega, -\omega)$ are rational functions of ω .
- The functions $\Gamma_{nn}(\omega, -\omega)$ have no repeated poles in the ω -plane.

Then the choice of $H(j\omega)$ which is linear, causal (all poles in the left half s -plane) and minimizes the expansion on the left side of (6.1) for $I(j\omega) = 1$ (the filtering case) is given by:

$$\hat{H}(j\omega) = \hat{H}(s) \Big|_{s=j\omega} = \left[\frac{F_1(s)}{[\bar{\Gamma}_{yy}(s)]^+} \right] \Big|_{s=j\omega}$$

where

$$\bar{\Gamma}_{yy}(s) = [\bar{\Gamma}_{yy}(s)]^+ [\bar{\Gamma}_{yy}(s)]^- = \bar{\Gamma}_{ss}(s) + \bar{\Gamma}_{nn}(s)$$

$$\bar{\Gamma}_{ss}(s) = \Gamma_{ss}(\omega, -\omega) \Big|_{s=j\omega}$$

$$\bar{\Gamma}_{nn}(s) = \Gamma_{nn}(\omega, -\omega) \Big|_{s=j\omega}$$

$$\frac{\bar{\Gamma}_{ss}(s)}{[\bar{\Gamma}_{yy}(s)]^-} = F_1(s) + F_2(-s)$$

$F_1(s)$ denotes the factored portion that is analytic in the right one-half s -plane

$F_2(-s)$ denotes the factored portion that is analytic in the left one-half s -plane

$[]^+$ denotes that portion of $[]$ having poles and zeroes in the left one-half s -plane only

$[]^-$ denotes that portion of $[]$ having poles and zeroes in the right one-half s -plane only.

Proof

The proof of this theorem is given in reference [4] as theorem 2.

VII. Application of the Result

The result given by the theorem indicates that for nonstationary processes with two-dimensional Fourier transforms it is possible to arrive at an $H(j\omega)$ for which the integral of the mean square estimation error is minimized. Clearly, the class of nonstationary processes for which:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |R_{ss}(t_1, t_2)| dt_1 dt_2 < \infty \quad (7.1)$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |R_{nn}(t_1, t_2)| dt_1 dt_2 < \infty \quad (7.2)$$

may not be all that large, but it is quite possible that some of the processes encountered in practical situations may be approximated by such processes.

If one considers the nature of those processes that arise from random initial conditions applied to linear dynamical systems for instance, one finds autocorrelation functions of the form that satisfy (7.1) and (7.2). The following example demonstrates the application of the theorem to a three-dimensional system with scalar observations and white nonstationary disturbances.

Example

In this example, it is assumed that the system is described by a vector matrix differential equation. The observation is a scalar valued function of the system's states with additive white nonstationary noise.

Assumptions:

Vector System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Scalar Observation

$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + v$$

Noise Processes

$$E[u(t)] = E[v(t)] = 0$$

$$E[u(t_1)u(t_2)] = e^{-0.1(|t_1|+|t_2|)} \delta(t_1-t_2)$$

$$E[v(t_1)v(t_2)] = e^{-0.5(|t_1|+|t_2|)} \delta(t_1-t_2)$$

For this system the transfer function is just

$$G(j\omega) = \frac{2}{(j\omega + 2)(j\omega + 3)(j\omega + 1)}$$

The two-dimensional transform of the signal autocorrelation is

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20 Abstract

→ an estimator depending in a prescribed way on the two-dimensional Fourier transforms of the state and observation noise. The resulting estimator is causal, linear, and time invariant. It is similar in some respects to the Wiener filter that can be derived under the assumption of stationary state and observation noise processes. The estimator's usefulness is limited by the requirement that the observations be scalar, and the nonstationary processes have Fourier transformable autocorrelation functions. ^{te}

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$$\Gamma_{ss}(\omega_1, \omega_2) = \Gamma_{uu}(\omega_1, \omega_2) G(j\omega_1) G(j\omega_2)$$

$$= \left[\frac{4(0.1)}{4(0.1)^2 + (\omega_1 + \omega_2)^2} \right] \left[\frac{2}{(j\omega_1 + 2)(j\omega_1 + 3)(j\omega_1 + 1)} \right]$$

$$\left[\frac{2}{(j\omega_2 + 2)(j\omega_2 + 3)(j\omega_2 + 1)} \right]$$

The observation noise transform is:

$$\Gamma_{nn}(\omega_1, \omega_2) = \Gamma_{vv}(\omega_1, \omega_2) = \frac{4(0.5)}{4(0.5)^2 + (\omega_1 + \omega_2)^2}$$

The average energy spectra becomes:

$$\Gamma_{ss}(\omega, -\omega) = \frac{4}{(0.1)(\omega^2 + (2)^2)(\omega^2 + (3)^2)(\omega^2 + 1)}$$

$$\Gamma_{nn}(\omega, -\omega) = \frac{1}{0.5}$$

Using these spectra the optimal filter, as indicated by the eq. 2 is just

$$\hat{R}(s) = \frac{0.1436 (s^2 + 6.071s + 10.324)}{(s + 3.0718)(s^2 + 3.0718s + 2.4359)}$$

VIII. Conclusions

This paper has discussed a technique whereby one can synthesize a causal, linear, time-invariant estimator that is optimal for restricted types of nonstationary processes. The results are applicable to linear, time-invariant systems (driven by nonstationary state noise) for which scalar observations are made in the presence of additive nonstationary noise. The resulting estimator is similar in some respects to the Wiener filter that can be derived under the assumptions of stationary state and observation noise processes. This similarity is reflected in the fact that the power spectra of the stationary processes are replaced by the average energy spectra of the nonstationary processes. The estimator's usefulness is limited by the requirement that the observations be scalar, and the nonstationary processes have Fourier transformable autocorrelation functions.

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